L^p Operator Algebras.

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Abstract

The term C^*-algebra was introduced by I. E. Segal in 1947 to describe norm-closed subalgebras of the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. A natural generalization is to replace the Hilbert space (which is an L^2- space) with an L^p-space, where $p \in [1, \infty]$. This gives rise to the study of L^p operator algebras. For $p \neq 2$, we don't have an inner product and therefore the geometry of an L^p space is much more complicated than the one of a Hilbert space. This makes the study of L^p operator algebras more complicated than the one of C^*-algebras. Nevertheless, there are some C^* results and constructions that still hold for L^p operator algebras.

In this talk I will define what an L^p operator algebra is and give several examples. I plan to spend most of the time doing this and the only prerequisite will be basic measure theory: anyone who knows what an L^p space is should be able to follow most of the talk. I will then talk about an L^p analog of the Cuntz algebras that comes from looking at the Leavitt algebras. Time permitting, I will talk about the full and reduced Crossed products of an L^p operator algebra A when we have an isometric action of a second countable locally compact group G on A.

1 Definition and Examples

Definition 1.1. Let A be a Banach algebra, and let $p \in [1, \infty]$. We say that A is an L^p **operator algebra** if there is a measure space $(X, \mathcal{B}, \mu)$ such that A is isometrically isomorphic to a norm closed subalgebra of $\mathcal{L}\big(L^p(X, \mu)\big)$.

We now give several examples of L^p operator algebras.

Example 1.2. For any $(X, \mathcal{B}, \mu)$ and $p \in [1, \infty]$, we trivially have that $\mathcal{L}\big(L^p(X, \mu)\big)$ is an L^p operator algebra.

Example 1.3. For any $(X, \mathcal{B}, \mu)$ and $p \in [1, \infty]$, the algebra $\mathcal{K}\big(L^p(X, \mu)\big)$ of compact operators on $L^p(X, \mu)$ is an L^p operator algebra.

Example 1.4. Any C^*-algebra is an L^2 operator algebra. However, a general L^2 operator algebra is a not necessarily a self-adjoint algebra.

Example 1.5. Let $p \in [1, \infty]$, $n \in \mathbb{Z}_{<0}$ and endow $\mathbb{C}^n$ with the usual p-norm:

$$\|z\|_p = \begin{cases} (\sum_{j=1}^n |z_j|^p)^{1/p} & \text{if } p \in [1, \infty) \\ \max_{j=1,\dots,n} |z_j| & \text{if } p = \infty \end{cases}$$

Then, if we equip M_n, the set of $n \times n$ complex matrices, with the operator norm we get an L^p operator algebra that is isometrically isomorphic to $\mathcal{L}(\ell^p(\{1, \dots, n\}))$. To emphasize the dependence on the p-norm, this space is denoted by M_n^p.

Example 1.6. For $j, k \in \{1, \dots, n\}$, let $e_{j,k} \in M_n^p$ be the matrix whose only non-zero entry is the entry (j, k) which is equal to 1. Then, the set of upper triangular matrices

$$T_n^p = \operatorname{span}\{e_{j,k} : 1 \leq j \leq k \leq n\}$$

is a subalgebra of M_n^p, which is also an L^p operator algebra.

Example 1.7. Let $p \in [1, \infty)$ and let X be a compact topological space. Suppose that μ is a regular Borel measure on X such that $\mu(U) > 0$ for every open set $U \subseteq X$ (we can always find such measure when X is compact metrizable and in some other cases). Then, $C(X)$ is an L^p operator algebra. Indeed, Let $\varphi : C(X) \to \mathcal{L}\big(L^p(X, \mu)\big)$ be given by

$$(\varphi(f)\xi)(x) := f(x)\xi(x)$$

We have to check that $\varphi(f)\xi \in L^p(X, \mu)$ for any $\xi \in L^p(X, \mu)$ and that φ is an injective isometric homomorphism. The former follows directly from

$$\int_X |f(x)\xi(x)|^p d\mu(x) \leq \|f\|_\infty^p \|\xi\|_p^p$$

For the latter, it's clear that φ is a homomorphism and since $\xi \equiv 1 \in L^p(X, \mu)$ (because $\mu(X) < \infty$), we also have that φ is injective. We now only need to prove that φ is isometric. Well, we already saw that $\|\varphi(f)\xi\|_p^p \leq \|f\|_\infty^p \|\xi\|_p^p$, so $\|\varphi(f)\| \leq \|f\|_\infty$. For the reverse inequality, assume that $\|f\|_\infty > 0$ (otherwise the desired inequality is trivial). For any $c \geq 0$ with $c < \|f\|_\infty$ we have that

$$U := \{x \in X : |f(x)| > c\}$$

is an open set and therefore $\mu(U) > 0$. Furthermore, notice that

$$\|\varphi(f)\chi_U\|_p^p = \int_U |f|^p d\mu > c^p \mu(U) = c^p \|\chi_U\|_p^p$$

Hence,

$$\|\varphi(f)\| \geq \frac{\|\varphi(f)\chi_U\|_p}{\|\chi_U\|_p} > c,$$

and since this holds for any c with $c < \|f\|_\infty$, it follows that $\|\varphi(f)\| \geq \|f\|_\infty$ as desired.

Example 1.8. Let $p \in [1, \infty]$ and let X be a locally compact topological space. Then $C_0(X)$, with the usual supremum norm, is an L^p operator algebra. To see this, let ν be counting measure on X and define $\varphi : C_0(X) \to \mathcal{L}\big(L^p(X, \nu)\big)$ by

$$(\varphi(f)\xi)(x) := f(x)\xi(x)$$

One checks that φ is an isometric bijection from $C_0(X)$ to a norm closed subalgebra of $\mathcal{L}(L^p(X, \nu))$.

The maps φ used in the previous two examples are special cases of representations.

Definition 1.9. **A representation of a Banach algebra** A **on a Banach space** E is a continuous homomorphism $\varphi : A \to \mathcal{L}(E)$. We say that a representation φ is non-degenerate if

$$\varphi(A)E := \operatorname{span}\{\varphi(a)\xi : a \in A, \xi \in E\}$$

is dense in E.

When studying L^p operator algebras we are interested in representations on L^p spaces. In fact, an L^p operator algebra is a Banach algebra for which there is an isometric representation on $L^p(X, \mu)$ for some measure space $(X, \mathcal{B}, \mu)$.

Example 1.10. Let A be the subalgebra of T_2^p generated by $e_{1,2}$. This is an L^p operator algebra. We claim that A does not admit non-degenerate representations. Indeed, assume that $\varphi : A \to \mathcal{L}(E)$ is a representation on **any** non-zero Banach space E. Since $\varphi(e_{1,2}) = 0$ it follows that if $\xi \in E$, then $\varphi(e_{1,2})\xi \in \ker(\varphi(e_{1,2}))$. This gives of course that $\varphi(A)E \subset \ker(\varphi(e_{1,2}))$. We have now two cases:

(1) $\varphi(e_{1,2}) \neq 0$. Here, $\ker(\varphi(e_{1,2}))$ is a proper subset of E (as there is a $\xi \in E$ for which $\varphi(e_{1,2})\xi \neq 0$) which is also closed. Hence, $\varphi(A)E$ cannot be dense in E.

(2) $\varphi(e_{1,2}) = 0$. Here, $\varphi(A)E = \{0\}$, so again it cannot be dense in E.

Example 1.11. As a final example we show how one can get new L^p operator algebras from old ones. For $p \in [1, \infty)$, if $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ are measure spaces, there is an L^p tensor product $L^p(X, \mu) \otimes_p L^p(Y, \nu)$, which can be canonically identified with $L^p(X \times Y, \mu \times \nu)$ via $(\xi \otimes \eta)(x, y) = \xi(x)\eta(y)$. Moreover:

- Let $(X_j, \mathcal{B}_j, \mu_j)$ and $(Y_j, \mathcal{C}_j, \nu_j)$ be measure spaces for $j = 1, 2$. If $a \in \mathcal{L}(L^p(X_1, \mu_1), L^p(X_2, \mu_2))$ and $b \in \mathcal{L}(L^p(Y_1, \nu_1), L^p(Y_2, \nu_2))$, then there is $a \otimes b \in \mathcal{L}(L^p(X_1 \times Y_1, \mu_1 \times \nu_1), L^p(X_2 \times Y_2, \mu_2 \times \nu_2))$, which has the expected properties: bilinearity, $\|a \otimes b\| = \|a\|\|b\|$ and $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$.

Then, if $A_j \subset L^p(X_j, \mu_j)$ is a norm closed subalgebra for $j = 1, 2$. We define

$$A_1 \otimes_p A_2 \subset \mathcal{L}\big(L^p(X_1 \times X_2, \mu_1 \times \mu_2)\big)$$

as the closed linear span of $a_1 \otimes a_2$ for $a_1 \in A_1$ and $a_2 \in A_2$. Then $A_1 \otimes_p A_2$ is an L^p operator algebra.

2 Analogs of Cuntz Algebras

Let $n \geq 2$ be an integer and $\mathcal{H}$ an infinite dimensional separable Hilbert space. Then, there are elements $s_1, s_2, \ldots, s_n \in \mathcal{L}(\mathcal{H})$ such that

$$s_j^* s_j = 1 \quad \text{and} \quad \sum_{j=1}^{n} s_j s_j^* = 1 \tag{$\star$}$$

We define $\mathcal{O}_n$, the Cuntz algebra of order n, as $C^*(s_1, \ldots, s_n)$. In fact, the construction of $\mathcal{O}_n$ is independent of the Hilbert space $\mathcal{H}$ and the choice of isometries as long as the relations $(\star)$ are satisfied. The algebra $\mathcal{O}_n$ is a simple C^*-algebra and has the following universal property: If A is a unital C^*-algebra containing elements $a_1, \ldots, a_n$ such that

$$a_j^* a_j = 1 \quad \text{and} \quad \sum_{j=1}^{n} a_j a_j^* = 1,$$

then there is a unique $*$-homomorphism $\varphi : \mathcal{O}_n \to A$ such that $\varphi(s_j) = a_j$.

We will now introduce the L^p-Cuntz algebras $\mathcal{O}_n^p$ which are an L^p analog of $\mathcal{O}_n$. In fact, when $p = 2$, we have $\mathcal{O}_n^2 = \mathcal{O}_n$. For that matter we first need to introduce an algebraic object: the Leavitt complex algebras.

Definition 2.1. Let $n \geq 2$ be an integer. We define the **Leavitt algebra** L_n to be the universal complex algebra generated by elements $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n$ satisfying

$$t_k s_j = \delta_{j,k} \quad \text{and} \quad \sum_{j=1}^{n} s_j t_j = 1$$

There is a well defined norm on L_n that comes from a particular kind of algebraic representations of L_n on σ-finite L^p spaces. The completion of L_n with respect to this norm is the L^p-Cuntz algebra $\mathcal{O}_n^p$. The algebraic representations that we need are the so called spatial representations. We need a little background to define these representations.

Notation 2.2. For a measure space $(X, \mathcal{B}, \mu)$ we have the set $\mathcal{N}(\mu) := \{E \in \mathcal{B} : \mu(E) = 0\}$. We put

$$[E] := \{F \in \mathcal{B} : E \triangle F \in \mathcal{N}(\mu)\}$$

and define $\mathcal{B}/\mathcal{N}(\mu) := \{[E] : E \in \mathcal{B}\}$. It's an easy exercise to check that the classic set operations, $\cup, \cap, \setminus$, are well defined in $\mathcal{B}/\mathcal{N}(\mu)$. In fact, $\mathcal{B}/\mathcal{N}(\mu)$ is a Boolean algebra.

Definition 2.3. Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be measure spaces. A **measurable set transformation** is a map $S : \mathcal{B}/\mathcal{N}(\mu) \to \mathcal{C}/\mathcal{N}(\nu)$ such that whenever $E_1, E_2, \ldots \in \mathcal{B}$

(i) $S(\mathcal{N}(\mu)) = \mathcal{N}(\nu)$ (i.e. $S([\varnothing]) = [\varnothing]$)

(ii) $S([X] \setminus [E_1]) = [Y] \setminus S([E_1])$

(iii) $S(\bigcup_{j=1}^{\infty} [E_j]) = \bigcup_{j=1}^{\infty} S([E_j])$

Moreover, we define the range of S as

$$\operatorname{ran}(S) := \{F \in \mathcal{C} : [F] = S([E]) \text{ for some } E \in \mathcal{B}\}$$

Remark 2.4. One checks that $\operatorname{ran}(S)$ is a sub σ-algebra of $\mathcal{C}$ and that S is surjective if and only if $\operatorname{ran}(S) = \mathcal{C}$.

Notation 2.5. Whenever we encounter a measurable set transformation $S : \mathcal{B}/\mathcal{N}(\mu) \to \mathcal{C}/\mathcal{N}(\nu)$ there will be some abuse of notation

- We will write $S(E) = F$ whenever $S([E]) = [F]$.
- We write $S : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ instead of $S : \mathcal{B}/\mathcal{N}(\mu) \to \mathcal{C}/\mathcal{N}(\nu)$.

Notation 2.6. For a measure space $(X, \mathcal{B}, \mu)$ we denote by $L^0(X, \mu)$ to the space of complex valued measurable functions modulo functions that vanish a.e $[\mu]$. Then, if $[E] \in \mathcal{B}/\mathcal{N}(\mu)$, the function $\chi_{[E]} \in L^0(X, \mu)$ is actually the equivalence class of χ_E in $L^0(X, \mu)$.

Proposition 2.7. *(The pushforward on $L^0(X, \mu)$ induced by S) Let $S : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ be a measurable set transformation. Then there is a unique linear map $S_* : L^*(X, \mu) \to L^0(Y, \nu)$ characterized by*

$$S_*(\chi_{[E]}) = \chi_{S([E])}$$

Definition 2.8. For a measure space $(X, \mathcal{B}, \mu)$ we define $\operatorname{ACM}(\mathcal{B}, \mu)$ to be the set of measures on $\mathcal{B}$ that are absolutely continuous with respect to μ. That is, $\operatorname{ACM}(\mathcal{B}, \mu) := \{$ measures λ on $\mathcal{B} : \lambda \ll \mu\}$.

Proposition 2.9. *(The pullback from $\operatorname{ACM}(\mathcal{C}, \nu)$ induced by S) Let $S : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ be a measurable set transformation. Then there is a unique map $S^* : \operatorname{ACM}(\mathcal{C}, \nu) \to \operatorname{ACM}(\mathcal{B}, \mu)$ characterized by*

$$S^*(\lambda)(E) = \lambda(F)$$

whenever $S([E]) = [F]$.

We really need to push measures forward rather than pull them back. For this, we require $S : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ to be an injective measurable transformation. Then, we get a measurable trnasformation $S^{-1} : (Y, \operatorname{ran}(S), \nu|_{\operatorname{ran}(S)}) \to (X, \mathcal{B}, \mu)$. This allows us to define the **pushforward on** $\operatorname{ACM}(\mathcal{B}, \mu)$ **induced by** S as follows: $S_* : \operatorname{ACM}(\mathcal{B}, \mu) \to \operatorname{ACM}(\operatorname{ran}(S), \nu|_{\operatorname{ran}(S)})$ is given by $S_* := (S^{-1})^*$.

Lemma 2.10. *Let $p \in [1,\infty]$ and $S : (X,\mathcal{B},\mu) \to (Y,\mathcal{C},\nu)$ be an injective measurable set transformation such that $\nu|_{\mathrm{ran}(S)}$ is σ-finite, and let g be a measurable function on Y such that $|g| = 1$ a.e. $[\nu]$. If $s : L^p(X,\mu) \to L^p(Y,\nu)$ is given by*

$$s(\xi) := \left(\frac{dS_*(\mu)}{d\nu|_{\mathrm{ran}(S)}}\right)^{1/p} S_*(\xi)g$$

Then, s is an isometry: $\|s(\xi)\|_p = \|\xi\|_p$.

Definition 2.11. Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ be σ-finite measure spaces.

(1) A **spatial system** for $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ is a quadruple (E,F,S,g) in which $E \in \mathcal{B}$, $F \in \mathcal{C}$, $S : (E,\mathcal{B}|_E,\mu|_E) \to (F,\mathcal{C}|_F,\nu|_F)$ is a bijective measurable set transformation and $g : F \to \mathbb{C}$ a $\mathrm{ran}(S)$-measurable function such that $|g| = 1$ a.e. $[\nu|_F]$.

(2) If $p \in [1,\infty]$, a linear map $s : L^p(X,\mu) \to L^p(Y,\nu)$ is said to be a **spatial partial isometry** if there is a spatial system (E,F,S,g) such that

$$s(\xi) := \begin{cases} \left(\frac{dS_*(\mu|_E)}{d\nu|_{\mathrm{ran}(S)}}\right)^{1/p} S_*(\xi|_E)g & \text{on } F \\ 0 & \text{on } Y \setminus F \end{cases}$$

An important theorem by Lamperti states that for $p \in [1,\infty) \setminus \{2\}$, any isometry $s \in \mathcal{L}\big(L^p(X,\mu), L^p(Y,\nu)\big)$ is a spatial isometry. Lamperti's theorem has been really useful to prove many facts about L^p operator algebras.

Lemma 2.12. *Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ be σ-finite measure spaces. Let $p \in [1,\infty]$, and let (E,F,S,g) be a spatial system for $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$.*

(1) There is a unique spatial partial isometry $s \in \mathcal{L}\big(L^p(X,\mu), L^p(Y,\nu)\big)$ whose spatial system is (E,F,S,g).

(2) Furthermore, there is a unique spatial partial isometry $t \in \mathcal{L}\big(L^p(Y,\nu), L^p(X,\mu)\big)$ whose spatial system is $(F,E,S^{-1},(S^{-1})_*(g)^{-1})$. The element t is called the ***reverse of*** *s.*

Definition 2.13. Let $n \geq 2$ be an integer, let $p \in [1,\infty]$ and let $(X,\mathcal{B},\mu)$ be a σ-finite measure space. A unital algebra homomorphism $\rho : L_n \to \mathcal{L}\big(L^p(X,\mu)\big)$ is said to be **a spatial representation** if for each j, the operators $\rho(s_j)$ and $\rho(t_j)$ are spatial partial isometries, with $\rho(t_j)$ being the reverse of $\rho(s_j)$.

Definition 2.14. Let $n \geq 2$ be an integer, let $p \in [1,\infty]$, and let $\rho : L_n \to \mathcal{L}\big(L^p(X,\mu)\big)$ be a spatial representation of L_n on a σ-finite space $(X,\mathcal{B},\mu)$. We define a spatial L^p operator norm on L_n by setting

$$\|a\| = \|\rho(a)\|.$$

This norm is seen to be independent of the spatial representation ρ chosen. We define the L^p **Cuntz algebra** $\mathcal{O}_n^p$ to be the completion of L_n in the above norm.

Theorem 2.15. *Let $n \geq 2$ be an integer and let $p \in [1,\infty)$. Then*

(1) $\mathcal{O}_n^p$ is simple.

(2) $\mathcal{O}_n^2 = \mathcal{O}_n$.

(3) $\mathcal{O}_n^p$ has the same K-theory as when $p = 2$: $K_0(\mathcal{O}_n^p) = \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_1(\mathcal{O}_n^p) = \{0\}$.

(4) $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \cong \mathcal{O}_2^p$ if and only if $p = 2$.

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